Generalized competing event model

Let \( n, k, \) and \( p \) be the number of observations, covariates, and mutually exclusive event types, respectively. Let \( z \) be the number cause-specific events, and \( p-z \) be the number of competing events. Let \( \mathbf{d} \) represent the \( k \times 1 \) vector of covariate values, and \( \mathbf{1}_m \) represent a \( m \times 1 \) vector of 1’s. Let \( i \) be an index of natural numbers ranging from 1 to \( p \). Let \( \lambda_{0i} \) represent the cause-specific hazard for event \( i \), and \( \lambda_0 = \Sigma \lambda_{0i} \) represent the hazard for any event, under a given set of experimental conditions.

We model the cause-specific hazard for event \( i \), under an alternative set of conditions as \( \lambda_{1i} = g(\mathbf{X}_i \beta_i) \lambda_{0i} \), for an invertible function \( g(\cdot) \), an \( n \times k \) data matrix \( \mathbf{X} \), and a \( k \times 1 \) vector of effect coefficients \( \beta_i \). The hazard for any event under the alternative set of conditions is \( \lambda_1 = \Sigma \lambda_{1i} = \Sigma g(\mathbf{X}_i \beta_i) \lambda_{0i} \) and the hazard ratio is expressed as:

\[
\frac{\lambda_1}{\lambda_0} = \frac{\Sigma g(\mathbf{X}_i \beta_i) \lambda_{0i}}{\Sigma \lambda_{0i}}
\]

in other words, the hazard ratio is a weighted average of the effects on the cause-specific hazards under the initial conditions. Here \( \beta \) is the \( k \times p \) coefficient matrix, with each element \( \beta_{v,w} \) representing the effect of covariate \( v \) on event \( w \). Note that under the assumption of effect homogeneity with respect to the cause-specific events, \( \beta_j = \beta_k = \beta \) for all \( j, k \in \{1, \ldots, p\} \), therefore:

\[
\frac{\lambda_1}{\lambda_0} = \frac{\Sigma g(\mathbf{X}_i \beta_i) \lambda_{0i}}{\Sigma \lambda_{0i}} = \frac{\Sigma g(\mathbf{X} \beta) \lambda_{0i}}{\Sigma \lambda_{0i}} = \frac{g(\mathbf{X} \beta) \Sigma \lambda_{0i}}{\Sigma \lambda_{0i}} = g(\mathbf{X} \beta).
\]

Let \( \mathbf{b}_i \) be a maximum (partial) likelihood estimator for \( \beta_i \) (e.g., using \( g(x) = e^x \)); alternatively, we can let \( \mathbf{b}_i \) represent an analogous maximum partial likelihood estimator
for sub-distribution hazards.\textsuperscript{2,3} Let \( B = [b_1 \ b_2 \ ... \ b_p] \) be the \( k \times p \) matrix of coefficients, with each element \( b_{v,w} \) of \( B \) representing the estimated effect of covariate \( v \) on event \( w \). Since columns of \( B \) are interchangeable, we can order the elements of \( B \) such that the first \( z \) vectors correspond to events of interest and the remaining \( p-z \) vectors correspond to competing events, i.e. \( B_{1,z} = [b_1 \ b_2 \ ... \ b_z] \) and \( B_{z,p} = [b_{z+1} \ b_{z+2} \ ... \ b_p] \), so \( B = [B_{1,z} \ B_{z,p}] \). Now using the data vector \( d \), we construct an individual risk score as follows:

\[
R = (d^T B_{z,p}) 1_{p-z} - (d^T B_{1,z}) 1_z
\]  

(3)

Note that under the assumption of effect homogeneity with respect to the cause-specific events, \( b_j = b_k = b \) for all \( j, k \in \{1,\ldots,p\} \), so \( R = cd^T b \) for some constant \( c \).

References